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SUFFICIENT CONDITIONS OF OPTIMALITY IN LINEAR PROBLEMS OF THE MATHEMATICAL THEORY OF OPTIMAL PROCESSES WITH PHASE CONSTRAINTS

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Iu. P. KRIVENKOV
(Moscow)

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Sufficient conditions of optimality in problems of the theory of optimal processes with phase constraints are formulated and proved. (see [1,2]). The problem of optimization is analyzed in the regular and singular cases in which control functions are expressed in terms of classical and generalized, in the meaning of Sobolev-Schwartz functions [3-5], respectively. Optimality conditions are formulated using systems of conjugate functions that also belong to certain classes of functions generalized in the meaning of Sobolev-Schwartz [6,7].

The latter makes it possible to obtain new forms of definition for conjugate functions and to formulate conditions of discontinuity, and also to extend the class of optimization problems and to simplify the class of conjugate functions [8-10].

Examples are given from the field of space navigation and of the theory of shells, which illustrate the distinctive feactures of the proposed approach [11].

1. The regular optimal problem. **Problem A.** Determine the n- and k-dimensional vector functions (columns) $x(t)$ and $u(t)$ along segment $[0, T]$ that satisfy the condition

$$\max\left\{\gamma x(T): \frac{dx}{dt} = Ax + Bu + a, \quad x(0) = c, \quad Pu \geqslant b, \quad Qx \geqslant d\right\}$$

where γ and c are constant n-dimensional (row, column) vectors; a, b and d are n-, s-, and m-dimensional vector functions (columns), and A, B, P, and Q are matrix functions of t in $[0, T]$ respectively, of orders $n \times n$, $n \times k$, $s \times k$, and $m \times n$.

Let $\mathbf{n}$ represent an n-dimensional vector with integral nonnegative components n_i, and p be a number or a symbol that satisfies the condition $1 \leqslant p \leqslant +\infty$. We introduce in the analysis the space $L_{n,p}^{\mathbf{n}}[0, T]$ of vector functions $x(t)$ of dimension n, each of whose components in measurable in $[0, T]$ and has the n_i-th derivative generalized in Sobolev's meaning and belonging to $L_p(0, T)$.

We seek the solution $x(t)$ and $u(t)$ of problem A in the class $x(t) \in L_{n,p}^{\mathbf{n}}[0, T]$, $u(t) \in L_{k,p}^{\mathbf{k}}[0, T]$ where $n_i \geqslant 1$, and $k_j = 0$, and assume that for any such $x(t)$ and $u(t)$ the inclusions

$$A(t)x(t), \quad B(t)u(t), \quad a(t) \in L_{n,p}^{\mathbf{n}-e}[0, T]$$

$$P(t)u(t), \quad b(t) \in L_{s,p}^{\mathbf{s}}[0, T]; \quad Q(t)x(t), \quad d(t) \in L_{m,p}^{\mathbf{m}}[0, T]$$

$$\mathbf{e} = (1, 1, \ldots, 1), \quad s_i = 0, \quad m_j \geqslant 1$$

are valid.

We assume that the equation $dx / dt = Ax + By + a$ and the inequality $Pa \geqslant b$ are on the average satisfied.

2. The space of generalized functions. Let us consider space D of vector functions (columns) $v(t)$ of dimension n with infinitely differentiable components $v_i(t)$, that vanish in some neighborhoods of points $t = 0$ and $t = T$.

Let $\mathbf{n}$ be a vector of dimension n with integral components n_i and $1 \leqslant p \leqslant +\infty$. We denote by $L^{\mathbf{n}}_{n,p}[0, T]$ the space of generalized vector functions $\varphi(t)$ the space of linear functionals $(\varphi(t), v(t))$ determinate in D_n and represented in the form

$$(\varphi(t), v(t)) = \sum_{i=1}^{n} \Big(\int_0^T \varphi_i^\circ(t) v_i(t)\, dt + \sum_{j=1}^{-n_i} \int_0^T \varphi_i^j(t) \frac{d^j}{dt^j} v_i(t)\, dt \Big)$$

where $\varphi_i^\circ(t) \in L_p(0, T)$ when $n_i \leqslant 0$; $\varphi_i^\circ(t) \in L^{n_i}_{1,p}[0, T]$ when $n_i > 0$, $\varphi_i^j(t) \in L_p(0, T)$ when $n_i < 0$, and $\varphi_i^j(t) = 0$ when $n_i \geqslant 0$.

3. Properties of generalized functions.

1°. If $\varphi(t) \in L^{\mathbf{n}}_{n,p}[0, T]$ and $\psi(t) \in L^{\mathbf{r}}_{n,p}[0, T]$, then the sum $\varphi(t) + \psi(t) \in L^{\mathbf{s}}_{n,p}[0, T]$, where $s_i \geqslant \max\{n_i, r_i\}$, is determined.

2°. If $\varphi(t) \in L^{\mathbf{n}}_{n,p}[0, T]$ and $A(t)$ is a matrix function of order $n \times k$ whose components are $a_{ij}(t) \in C^n[0, T]$, then the product $\varphi(t)A(t) \in L^{\mathbf{k}}_{k,p}[0, T]$, where $k_j \geqslant \min_i\{n_i\}$, is determined as

$$(\varphi(t)A(t), v(t) = (\varphi(t), A(t)v(t))$$

3° If $\varphi(t) \in L^{\mathbf{n}}_{n,p}[0, T]$, then the derivative $d\varphi(t) / dt \in L^{\mathbf{n}-\mathbf{e}}_{n,p}[0, T]$ is determined as

$$\Big(\frac{d\varphi(t)}{dt}, v(t)\Big) = -\Big(\varphi(t), \frac{dv(t)}{dt}\Big)$$

4°. Each functional $(\varphi(t), v(t))$ that determines $\varphi(t) \in L^{\mathbf{n}}_{n,p}[0, T]$ is represented in the form of the sum

$$(\varphi(t), v(t)) = \sum_{i=1}^{n} (\varphi_i(t), v_i(t))$$

of functionals $(\varphi_i(t), v_i(t))$ that determine $\varphi_i(t) \in L_{1,p}^{n_i}[0, T]$ and represent the i-th coordinates $\varphi(t)$.

5°. Each function $\varphi(t) \in L_{n,p}^{n}[0, T]$, where $\min_i \{n_i\} = n_0 < 0$, is represented by the sum of the regular $\varphi^r(t)$ and singular $\varphi^s(t)$ components of $\varphi(t)$, i.e. $\varphi(t) = \varphi^r(t) + \varphi^s(t)$. Each coordinate $\varphi_i^r(t)$ of the regular component $\varphi^r(t)$ is of the form

$$(\varphi_i^r(t), v_i(t)) = \int_0^T \varphi_i^0(t)\, v_i(t)\, dt$$

and on the average can be identified with function $\varphi_i^{(0)}(t)$ of that formula, while each coordinate $\varphi_i^s(t)$ of the singular component $\varphi^s(t)$ can be represented for $n_i < 0$ in the form

$$(\varphi_i^s(t), v_i(t)) = \int_0^T \Phi_i(t) \frac{d^{-n_i}}{dt^{-n_i}} v_i(t)\, dt$$

$$\Phi_i(t) = \varphi_i^{-n_i}(t) - \int_0^t \varphi_i^{-n_i-1}(t)\, dt + \ldots$$

$$(-1)^{-n_i-1} \int_0^t \ldots \int_0^t \varphi_i^1(t)\, (dt)^{-n_i-1}$$

and be considered as the n_i-th generalized derivative of function $(-1)^{n_i} \Phi_i(t) \in L_p(0, T)$.

6°. If $\varphi(t) \in L_{n,p}^{n}[0, T]$ and $u(t) \in L_{n,q}^{m}[0, T]$, where $n + m \geqslant (0, \ldots, 0)$, $1 \leqslant p \leqslant +\infty$ and $q = p/(p-1)$, the product $\varphi(t) u(t) \in L_{1,1}^{n^\circ}[0, T]$, where $n^\circ = \min\{\min_i\{n_i\}, \min_j\{m_j\}\}$, is determined in the form

$$(\varphi(t)\, u(t), v(t)) = \sum_{i=1 (m_i \geqslant 0)}^{n} (\varphi_i(t), u_i(t)\, v(t)) +$$

$$\sum_{i=1 (m_i < 0)}^{n} (u_i(t), \varphi_i(t)\, v(t))$$

7°. If $\varphi(t) \in L_{n,p}^{n}[0, T]$ and $u(t) \in L_{n,q}^{m}[0, T]$, where $n + m = e$, $1 \leqslant p \leqslant +\infty$ and $q = p/(p-1)$, the derivative $(d/dt) \cdot (\varphi(t) u(t)) \in L_{1,1}^{n^\circ-1}[0, T]$ with the properties

$$\frac{d}{dt}(\varphi(t)u(t)) = \frac{d\varphi(t)}{dt}u(t) + \varphi(t)\frac{du(t)}{dt}$$

is determined.

4. Semiorderliness of generalized functions. Let us consider in $[0, T]$ n closed sets $\overline{\omega}_i$. We assume that the totality of sets $\{\overline{\omega}_i\}$ belongs to class Ω, i.e. $\{\overline{\omega}_i\} \in \Omega$ if: a) there exists an $\varepsilon > 0$ such that for any $i \in \{1, \ldots, n\}$ the inclusion $\omega_i \subset [\varepsilon, T - \varepsilon]$ is valid; b) for any $i \in \{1, \ldots, n\}$ the internal part of set $\overline{\omega}_i$, i.e. $\omega_i = \operatorname{int} \overline{\omega}_i$ represents a finite number of isolated intervals, and c) the complement ω_i in $\overline{\omega}_i$, i.e. $\gamma_i = \overline{\omega}_i \setminus \omega_i$ can be represented in the form $\gamma_i = \gamma_i^- \cup \gamma_i^+ \cup \gamma_i^\circ$, where γ_i^- and γ_i^+ are sets of the left- and right- hand ends of intervals belonging to ω_i, and γ_i° is the set consisting of a finite number of points isolated from ω_i.

In the space $L^{\mathrm{n}}_{n,p}[0, T]$ of the generalized functions $\varphi(t)$, we introduce, using the set $\{\overline{\omega}_i\} \in \Omega$ the relation of semiorderliness $\varphi(t) \dot{\geqslant} 0$, and assume that $\varphi(t) \dot{\geqslant} 0$ if the following conditions are satisfied.

For any $i \in \{1, \ldots, n\}$, for which $n_i \geqslant 0$, the inequality $\varphi_i^\circ(t) \geqslant 0$ is valid in $[0, T]$ in the average when $n_i = 0$ and pointwise when $n_i > 0$.

For any $i \in \{1, \ldots, n\}$, for which $n_i \geqslant 2$ and for any $j \in \{1, \ldots, n_i - 1\}$ the inequalities

$$(-1)^j \frac{d^j}{dt^j}\varphi_i^\circ(t) \geqslant 0 \quad \text{in} \quad \gamma_i^-, \quad \frac{d^j}{dt^j}\varphi_i^\circ(t) \geqslant 0 \quad \text{in} \quad \gamma_i^+$$

$$[1 + (-1)^j]\frac{d^j}{dt^j}\varphi_i^{(0)}(t) \geqslant 0 \text{ in } \gamma_i^\circ$$

are valid

For any $i \in \{1, \ldots, n\}$, for which $n_i < 0$, any $k \in \{1, \ldots, -n_i\}$, and any function $v(t) \in L^{-k}_{1,q}[0, T]$, and $q = p/(p-1)$ that satisfy the relation $v(t) \dot{\geqslant} 0$ in that space, the inequality

$$\int_0^T \varphi_i^k(t)\frac{d^k}{dt^k}v(t)\,dt \geqslant 0$$

is valid.

Let us point out some of the properties of introduced relation.

1°. If $\varphi(t) \in L^{\mathrm{n}}_{n,p}[0, T]$ and $u(t) \in L^{\mathrm{m}}_{n,q}[0, T]$, where $\mathbf{n} + \mathbf{m} \geqslant (0, \ldots, 0)$, $1 \leqslant p \leqslant +\infty$, $q = p/(p-1)$, $\varphi(t) \dot{\geqslant} 0$ and $u(t) \dot{\geqslant} 0$, then $\varphi(t)u(t) \dot{\geqslant} 0$.

2°. Let $\delta^{(k)}(t - \tau) \in L^{k+1}_{1,p}[0, T]$ represent the k-th derivative of the δ-function at point τ. Then the relation $\alpha\delta^{(k)}(t - \tau) \dot{\geqslant} 0$, where $\alpha = \text{const}$, means that $\alpha \geqslant 0$, when k is even $\alpha \geqslant 0$ when $\tau \in \gamma^-$, $\alpha \leqslant 0$ when $\tau \in \gamma^+$, and $\alpha = 0$ when $\tau \in \gamma^\circ$, if k is odd.

5. Conditions of optimality for the regular optimality problem. Let us consider some admissible solutions $x^*(t)$, $u^*(t)$ of problem A, which we shall use for determining the totality of sets $\{\overline{\omega}_i\}$, where $i \in \{1, \ldots, m\}$, in the form $\overline{\omega}_i = \{t : Q_i(t)x^*(t) - d_i(t) = 0\}$, where $Q_i(t)x^*(t)$ and $d_i(t)$ are the i-th components of vector functions $Q(t)x^*(t)$ and $d(t)$. The totality of sets $\{\overline{\omega}_i\}$ is assumed to belong to the Ω class. Using the described above procedure we introduce the semiorderliness relations $g(t) \dot{\geqslant} 0$ and $\vartheta(t) \dot{\geqslant} 0$, respectively, for the spaces $L^{m}_{m,p}[0, T]$ and $L^{-m}_{m,q}[0, T]$, where $q = p/(p-1)$. Noting that the admissible solutions $x^*(t)$ and $u^*(t)$ satisfy the relation $Q(t)x^*(t) \dot{\geqslant} d(t)$, we can say that the admissible solution $x(t)$, $u(t)$ of problem A is close to the admissible solution $x^*(t)$, $u^*(t)$ with respect to semiorderliness $Qx \dot{\geqslant} d$, if $Q(t)x(t) \dot{\geqslant} d(t)$.

Theorem 1. If for some admissible solution $x^*(t)$, $u^*(t)$ of problem A there exist functions

$$\psi(t) \in L^{-n+e}_{n,q}[0, T], \quad \varepsilon(t) \in L^{-s}_{s,q}[0, T]$$

$$\vartheta(t) \in L^{-m}_{m,q}[0, T] \quad \left(q = \frac{p}{p-1}\right)$$

such that function $\psi(t)$ is regular and continuous in the neighborhoods of points $t = 0$ and $t = T$, and functions $\psi(t)$, $\varepsilon(t)$, and $\vartheta(t)$ satisfy system

$$d\psi/dt + \psi A + \vartheta Q = 0, \quad \psi(T) = \gamma, \quad \psi B + \varepsilon P = 0$$

$$\varepsilon \geqslant 0, \quad \vartheta(t) \dot{\geqslant} 0, \quad \varepsilon(Pu^* - b) = 0, \quad \vartheta(Qx^* - d) = 0$$

then $x^*(t)$, $u^*(t)$ is the optimal solution among the admissible solutions of problem A that are close to $x^*(t)$, $u^*(t)$ with respect to the semiorderliness $Qx \dot{\geqslant} d$.

Proof. Let us assume that an admissible solution $x(t)$, $u(t)$ which satisfies the inequality $\gamma x(T) - \gamma x^*(T) = \varepsilon > 0$ exists in the neighborhood of solution $x^*(t)$, $u^*(t)$ admissible with respect to the semiorderliness $Qx \dot{\geqslant} d$. For functions $\psi(t) \in L^{-n+e}_{n,q}[0, T]$ and $\bar{x}(t) = x(t) - x^*(t) \in L^{n}_{n,p}[0, T]$ the following product is determined:

$$-\frac{d\psi}{dt}\bar{x}(t) \in L^{n^*}_{1,1}[0, T] \quad (n^* = \min_i\{-n_i\})$$

Because $\psi(t)$ satisfies the equation $-d\psi/dt = \psi A + \vartheta Q$ this product is equal to $\psi A\bar{x} + \vartheta Q\bar{x}$, which means that the equality

$$-\frac{d\psi}{dt}\bar{x} = \psi A\bar{x} + \vartheta Q\bar{x}$$

is valid.

Since

$$\psi B + \varepsilon P = 0, \quad \varepsilon (Pu^* - b) = 0, \quad \vartheta (Qx^* - d) = 0$$
$$d\bar{x} / dt = A\bar{x} + B\bar{u} \quad (\bar{u}(t) = u(t) - u^*(t))$$

that equality assumes the form

$$- d / dt (\psi\bar{x}) = \varepsilon (Pu - b) + \vartheta (Qx - d)$$

Because $\varepsilon (t) \geqslant 0, Pu + b \geqslant 0, \vartheta (t) \geqslant 0, Qx - d \geqslant 0,$ the latter equality generates the inequality $(d / dt)(\psi\bar{x}) \leqslant 0$.

In conformity with conditions of the theorem function $\psi (t)$ is regular in the neighborhoods $[0, \sigma_1]$ and $[T - \sigma_1, T]$, where $\sigma_1 > 0$. We take an arbitrary $\sigma \in (0, \sigma_1]$ and construct function $v_\sigma (t) \in D_1$ of the form $v_\sigma (t) = 0$ in $[0, \sigma / 2]$ and $[T - \sigma / 2, T]$, $v_\sigma (t) = 1$ in $[\sigma, T - \sigma]$, $v_\sigma (t) = \eta (2t / \sigma - 1)$ in $[\sigma / 2, \sigma]$, and $v_\sigma (t) = \eta (2 (T - t) / \sigma - 1)$ in $[T - \sigma, T - \sigma / 2]$, where

$$\eta (\tau) = \frac{1}{M} \int_0^\tau \exp \left(- \frac{1}{t (1 - t)}\right) dt, \quad M = \int_0^T \exp \left(- \frac{1}{t (1 - t)}\right) dt > 0$$

Then

$$\frac{d}{dt} (\psi\bar{x}) = - \sum_{i=1}^n \left(\int_{\sigma/2}^{\sigma} + \int_{T-\sigma}^{T-\sigma/2} \right) \psi_i^\circ (t) \bar{x}_i (t) \frac{dv_\sigma}{dt} dt$$

Functions $\psi (t)$ and $\bar{x} (t)$ are continuous in the neighborhood of points $t = 0$ and $t = T$, hence there exists for any $\mu > 0$ a $\sigma_2 > 0$ such that

$$\psi_i^\circ (t)\bar{x}_i (t) = \psi_i^\circ (0)\bar{x}_i (0) + \xi (t), \quad | \xi (t) | \leqslant \mu, \ \forall\ t \in [0, \sigma_2]$$
$$\psi_i^\circ (t)\bar{x}_i (t) = \psi_i^\circ (T)\bar{x}_i (T) + \zeta (t), \quad | \zeta (t) | \leqslant \mu, \ \forall\ t \in [T - \sigma_2, T]$$

We select $\sigma = \min \{\sigma_1, \sigma_2\}$ and obtain the inequality

$$- \sum_{i=1}^n (\psi_i^\circ (0) \bar{x}_i (0) - \psi_i^\circ (T) \bar{x}_i (T)) - \sum_{i=1}^n \left(\int_{\sigma/2}^{\sigma} \xi (t) \frac{d}{dt} v_\sigma (t)\, dt + \int_{T-\sigma}^{T-\sigma/2} \zeta (t) \frac{d}{dt} v_\sigma (t)\, dt \right) \leqslant 0$$

which owing to $\bar{x} (0) = 0$ and $\psi (T) = \gamma$ implies the inequality

$$\gamma\bar{x} (T) \leqslant \sum_{i=1}^n \left(\int_{\sigma/2}^{\sigma} \xi (t) \frac{d}{dt} v_\sigma (t)\, dt + \int_{T-\sigma}^{T-\sigma/2} \zeta (t) \frac{d}{dt} v_\sigma (t)\, dt \right)$$

Taking into account the estimates

$$\left|\frac{d}{d\tau}\eta(\tau)\right| \leqslant \frac{e^{-4}}{M}, \quad \left|\frac{d}{dt}v_\sigma(t)\right| \leqslant \frac{2e^{-4}}{\sigma M}$$

and selecting $\mu = Me^4\varepsilon / (4n)$, we obtain the inequality $\gamma\bar{x}(T) \leqslant \varepsilon / 2$ which contradicts the previously assumed inequality $\gamma\bar{x}(T) = \varepsilon > 0$. The theorem is proved.

Example 1. Determine on segment [0, 12] $x(t) \in L_{1,2}^2[0, 12]$, $y(t) \in L_{1,2}^1[0, 12]$ and $u(t) \in L_{1,2}^\circ[0, 12]$ which satisfy the condition

$$\max\left\{\int_0^{12} x\,dt:\ \frac{dx}{dt} = y,\quad \frac{dy}{dt} = -u,\ x(0) = 0,\ y(0) = -2,\quad |u| \leqslant 1,\ x \leqslant a(t)\right\}$$

$$a(t) = (t-10)^3(3t-14)/64 - 2$$

Let us consider the following admissible solutions: $u^*(t) = 1$ in the intervals (2, 4) and (8, 10); $u^*(t) = -1$ in (0, 2), (4, 8) and (11, 12), and $u^*(t) = -(d^2/dt^2)\,a(t)$ in (10, 11). For the stated problem and admissible solution the combination of point $t = 6$ and segment [10, 11] represent the set ω, while the separate points $t = 10$, $t = 11$, and $t = 6$ represent, respectively, the sets γ^-, γ^+, and γ°.

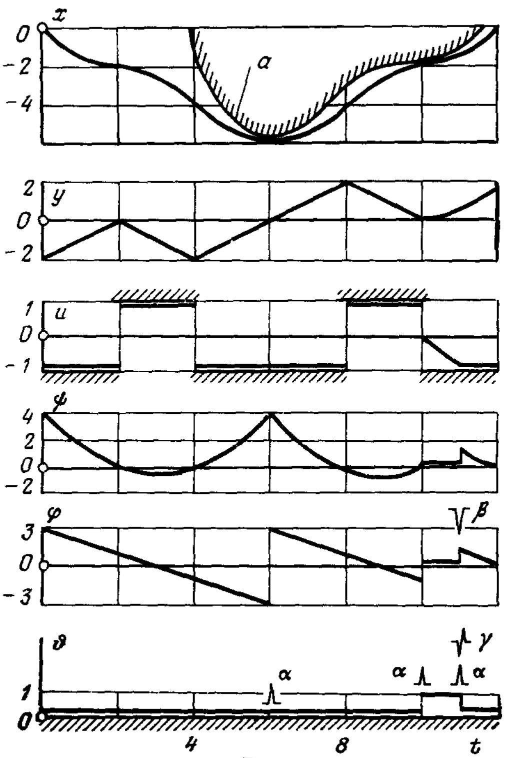

Fig. 1

Condition $g(t) = a(t) - x(t) \geqslant 0$ implies that

$$g(t) \geqslant 0, \quad \frac{d}{dt}g(t)\Big|_{t=10} \leqslant 0, \quad \frac{d}{dt}g(t)\Big|_{t=11} \geqslant 0$$

The classes of conjugate functions are defined as follows:

$$\varphi(t) \in L_{1,2}^{-1}[0,12],$$
$$\psi(t) \in L_{1,2}^\circ[0,12]$$
$$\varepsilon_1(t),\ \varepsilon_2(t) \in L_{1,2}^\circ[0,12],$$
$$\vartheta(t) \in L_{1,2}^{-2}[0,12]$$

and the conjugate system of conditions for these functions are of the form

$$d\varphi/dt = \vartheta(t) - 1, \quad d\psi/dt = -\varphi(t),$$
$$\varphi(12) = 0, \quad \psi(12) = 0$$
$$\psi(t) = \varepsilon_1(t) - \varepsilon_2(t),$$
$$\varepsilon_1(t)(1 + u^*(t)) = 0,$$
$$\varepsilon_2(t)(1 - u^*(t)) = 0$$
$$\vartheta(t)(a(t) - x^*(t)) = 0,\ \varepsilon_1(t) \geqslant 0,$$
$$\varepsilon_2(t) \geqslant 0, \quad \vartheta(t) \gtrless 0$$

If function $\vartheta(t)$ is of the form $\vartheta(t) = \vartheta^*(t) + \alpha\delta(t-\tau) + \beta\delta'(t-\tau)$, where $\vartheta^*(t) \in L_{1,2}{}^{\circ}[0,\ 12]$ and $\tau \in \gamma$, then condition $\vartheta(t) \geqslant 0$ means that: $\vartheta^*(t) \geqslant 0$; $\alpha \geqslant 0$, and $\beta \geqslant 0$ when $\tau = 10$, $\beta \leqslant 0$ when $\tau = 11$, and $\beta = 0$ when $\tau = 6$.

Using the notation $\Pi_\alpha{}^\beta(t) = \theta(t-\alpha) - \theta(t-\beta)$, where $\theta(t)$ is the characteristic function of set $\{t\colon\ t \geqslant 0\}$, we formulate the solution of the conjugate system $\vartheta(t) = \Pi_{10}{}^{11}(t) + 6\delta(t-6) + \delta(t-10) + \delta(t-11) - {}^1\!/_2\delta'(t-11)$ as follows: $\varphi(t) = 3 - t$ in $[0, 6]$, $\varphi(t) = 9 - t$ in $(6, 10)$, $\varphi(t) = 0$ in $(10, 11)$, $12 - t - {}^1\!/_2\delta(t-11)$ in $[11, 12]$, $\psi(t) = {}^1\!/_2(9-t)^2 - {}^1\!/_2$ in $[0, 6)$, $\psi(t) = 0$ in $(10, 11)$, $\psi(t) = {}^1\!/_2(t-12)^2$ in $(11, 12)$, (see Fig. 1, where α, β, and γ denote the δ-function, the δ-function with a negative coefficient, and the derivative of the latter, respectively).

In conformity with Theorem 1 the admissible solution $x^*(t), y^*(t), u^*(t)$ is the optimal solution for the Example 1 among solutions that by the semiorderliness condition $x(t) - a(t) \geqslant 0$ are close to it.

Note that the proposed here form of optimality conditions are simpler and more convenient than the conventionally used in that they neither stipulate the consideration and analysis of discontinuities of Hamiltonian $H(t)$ besides function $\psi(t)$, nor the introduction of additional constraints in the problem optimality conditions that are obtained by the first and second differentiation of functions and define and phase constraint $x(t) - a(t) \geqslant 0$ of the form

$$y(t) \geqslant \frac{d}{dt}a(t), \quad u(t) \leqslant -\frac{d^2}{dt^2}a(t)$$

Furthermore they do not require the introduction of additional conjugate functions or measures related to such constraints, and avoid the necessity of determining at the phase boundary $[10, 11] = \{t : x^*(t) = a(t)\}$ the values of functions $\psi(t)$ and $\varphi(t)$ by the fairly complex and cumbersome system of equations

$$\frac{d\varphi}{dt} = -\frac{\partial H}{\partial x} + \frac{1}{D}\frac{\partial H}{\partial u}D_x, \quad \frac{d\psi}{dt} = -\frac{\partial H}{\partial y} + \frac{1}{D}\frac{dH}{\partial u}D_y$$

where D, D_x, and D_y are some determinants.

6. The singular optimal problem. Problem B. Using condition

$$\max\ \{\gamma x(T)\colon dx/dt = Ax + Bu + a,\ x(0) = c,\ Pu \geqslant b,\ Qx \geqslant d\}$$

in which for any $x(t)$ and $u(t)$ from the admissible class of vector functions

$$A(t)x(t),\ \ B(t)u(t),\ a(t) \in L_{n,p}^{n-e}[0,\ T]$$
$$P(t)u(t),\ \ b(t) \in L_{s,p}{}^{s}[0,\ T];\quad Q(t)x(t),\ \ d(t) \in L_{m,p}{}^{m}[0,\ T]$$
$$(-1 \leqslant s_i \leqslant 0,\ m_j \geqslant 0)$$

determine on segment $[0,\ T]$ functions $x(t) \in L_{n,p}{}^{n}[0,\ T]$ and $u(t) \in L_{k,p}{}^{k}[0,\ T]$, where $1 \leqslant p \leqslant +\infty$, $n_i \geqslant 0$, $-1 \leqslant k_j \leqslant 0$ and function

$x(t)$ is continuous in the neighborhoods of points $t = 0$ and $t = T$.

The equation $dx / dt = Ax + Bu + a$ is to be understood in the defined above generalized sense. The inequality $h(t) = Pu - b \geqslant 0$ in which $h(t) \in L_{s,p}^{s}[0, T]$ is also to be understood in the above generalized sense, i.e. if $s_i = 0$, then $h_i(t) \geqslant 0$ in the average in $[0, T]$, and if $s_i = -1$, then for any function $v(t) \in D_1$, where $v(t) \geqslant 0$ the inequality $(h_i(t), v(t)) \geqslant 0$ is satisfied. The inequality $g(t) = Qx - d \geqslant 0$ in which $g(t) \in L_{m,p}^{m}[0, T]$ and $m_i \geqslant 0$ indicates the pointwise fulfilment of inequality $Q_i x(t) - d_i \geqslant 0$ when $m_i \geqslant 1$, and for $m_i \geqslant 1$ the latter is in the average satisfied,

Let us consider some admissible solution $x^*(t)$, $u^*(t)$ of problem B and as previously, determine the sets $\overline{\omega}_i = \{t : Q_i x^*(t) - d_i(t) = 0\}$ and γ_i^-, γ_i^+, and γ_i°, assuming that $\{\overline{\omega}_i\} \in \Omega$. We determine for the functions

$$g(t) \in L_{m,p}^{\mathrm{m}}[0, T], \quad \vartheta(t) \in L_{m,q}^{-\mathrm{m}}[0, T] \quad \left(q = \frac{p}{p-1}\right)$$

the respective semiorderliness relations $g(t) \dot{\geqslant} 0$ and $\vartheta(t) \dot{\geqslant} 0$; using the first of these we determine the condition of closeness of the admissible solution $x(t)$, $u(t)$ of problem B to the admissible solution $x^*(t)$, $u^*(t)$.

7. Optimality conditions for the singular optimal problem. **Theorem 2.** If some admissible solution $x^*(t)$, $u^*(t)$ of problem B there exists functions

$$\psi(t) \in L_{n,q}^{-\mathrm{n+e}}[0, T], \quad \varepsilon(t) \in L_{s,q}^{\mathrm{s}}[0, T]$$

$$\vartheta(t) \in L_{m,q}^{-\mathrm{m}}[0, T] \quad \left(q = \frac{p}{p-1}\right)$$

such that function $\psi(t)$ is regular and continuous in the neighborhood of points $t = 0$ and $t = T$, and functions $\psi(t)$, $\varepsilon(t)$, and $\vartheta(t)$ satisfy the system

$$d\psi / dt + \psi A + \vartheta Q = 0, \quad \psi(T) = \gamma, \quad \psi B + \varepsilon P = 0$$

$$\varepsilon(t) \geqslant 0, \quad \vartheta(t) \geqslant 0, \quad \varepsilon(t)(Pu^*(t) - b) = 0$$

$$\vartheta(t)(Qx^*(t) - d) = 0$$

then $x^*(t)$, $u^*(t)$ is the optimal solution of problem B among the admissible solutions of that problem that are close to $x^*(t)$, $u^*(t)$ with respect to the semiorderliness $Qx - d \geqslant 0$.

The proof of Theorem 2 follows exactly that of Theorem 1.

Example 2. Using condition

$$\max\left\{\int_0^{14} \alpha(x) u(x)\, dx: \quad \frac{d^2y}{dx^2} = u(x), \quad y(0) = 0,\right.$$

$$\left. y(x) \leqslant 0, \quad y(x) \leqslant a(x), \quad y(x) \leqslant b(x), \quad y(x) \geqslant c(x)\right\}$$

$$\alpha(x) = 2|x-4| - 5, \quad a(x) = x - 2|x-2|^2$$
$$b(x) = 14 - x + 2(x-12)^2, \quad c(x) = 5 - {}^1/_6|x-4| - {}^1/_4|x-10|$$

determine on segment $[0, 14]$ $y(x)$ and $u(x)$.

An example of the application of this method in mechanics is the problem of determining the optimal profile of a shell with nonpositive curvature, covering object A, with phase constraints, using the curvature weighted mean.

We shall show that the admissible solution $y^*(x) = 7 - |x-7|$, $u^*(t) = -2\delta(x-7)$ is the optimal one for this example (see Fig. 2, where β denotes the δ-function with a negative coefficient).

The optimality problem in this example evidently reduces to problem B.

The conjugate system is of the form

$$\frac{d^2\psi}{dx^2} = \vartheta(x) - \lambda(x) - \mu(x), \quad \psi(0) = \psi(14) = \left.\frac{d\psi}{dx}\right|_{x=14} = 0$$
$$\psi(x) = \varepsilon(x) - \alpha(x), \quad \vartheta(x) \dot{\geqslant} 0, \quad \lambda(x) \dot{\geqslant} 0, \quad \mu(x) \dot{\geqslant} 0$$
$$\varepsilon(x) \geqslant 0, \quad \varepsilon(x)\, u^*(x) = 0, \quad \lambda(x)(y^*(x) - a(x)) = 0$$
$$\mu(x)(y^*(x) - b(x)) = 0, \quad \vartheta(x)(y^*(x) - c(x)) = 0$$

Since the equalities $y^*(x) = a(x)$, $y^*(x) = b(x)$, and $y^*(x) = c(x)$ are only valid at points $x = 2$, $x = 12$, and $x = 10$, respectively, at the remaining points $d^2\psi / dx^2 = 0$, and, since $u^*(x) \neq 0$ only at point $x = 7$, hence $\varepsilon(7) = 0$.

We select $\psi(x) = {}^9/_2\, x$ for $x \in [0, 2]$, $\psi(x) = 13 - 2x$ for $x \in [2, 10]$, $\psi(x) = {}^7/_2(x - 12)$ for $x \in [10, 12]$, and $\psi(x) = 0$ for $x \in [12, 14]$, and obtain the inequalities

$$\varepsilon(x) = \psi(x) + \alpha(x) \geqslant 0, \quad \lambda(x) = {}^{13}/_2\, \delta(x-2) \geqslant 0$$
$$\vartheta(x) = {}^{11}/_2 \delta(x-10) \geqslant 0, \quad \mu(x) = {}^7/_2\, \delta(x-12) \geqslant 0$$

and the equality $\varepsilon(7) = 0$, which by Theorem 2 proves the optimality of solution $y^*(x)$, $u^*(x)$ (see Fig. 3).

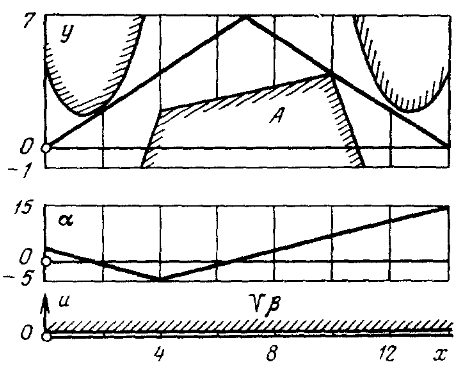

Fig. 2

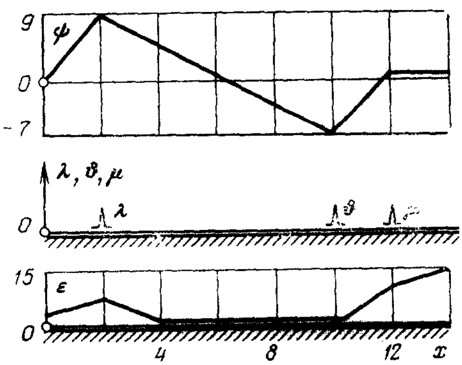

Fig. 3

This example shows that in many optimality problems which, in classical formulation do not have optimal solutions, may have such solutions with entirely real mechanical meaning in the generalized formulation. It also shows that in the considered above formulation of the optimization problem no assumption is made about any intial dislocation or the quantity of control function singularities.

8. **Concluding remarks.** The scheme for investigating solutions of optimization problem proposed in the present paper makes possible the following.

1°. Elimination of the restrictive consideration of the independent variable as the phase variable.

2°. Analyze problems in which control functions belong to spaces L_p or even D' [5].

3°. Derive conjugate systems without introducing additional constraints that are due to the differentiation of phase constraints, which considerably widens the dimension of the problem which results in the necessity to consider a fairly complex problem of variable structure [9].

4°. The admission of cases in which the basic conjugate function $\psi(t)$ may vanish at the phase boundary, which eliminates the additional integration of the conjugate system of equations at the phase boundary [8].

5°. To investigate besides the considered here and in [6] sufficient conditions of optimality, also, the necessary conditions, including the following complex and generalized cases:

a) in the presence of phase constraints of general form (*) and

b) for problems of optimization with distributed parameters (problems with differential equations in partialderivatives [7, 10].

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